FAR BEYOND

MAT122

Chain Rule

Stony Brook University

Review – Composition of Function

Given: $f(x)=\sqrt{x}$ $g(x)=x^2+1$ find $f(g(x))$

plug *g(x)* into *f(x)* to get composed function:

$$=f(x^2+1)$$

$$=\sqrt{x^2+1}$$

then $g=x^2+1$ is inner function

and $f=\sqrt{x}$ is outer function

Given: $h(x)=(x^3-1)^{100}$ if $h(x)=f(g(x))$ determine $f(x)$ and $g(x)$

$g=x^3-1$ is inner function

so $f=x^{100}$ is outer function

Tip: Pick an *inner* function such that the *outer* function has a SIMPLE derivative.

Differentiating Composed Functions

must take the derivative of BOTH inner and outer functions

if $h(x)$ is in the format $f(g(x))$ then

Chain Rule

$(outer)' \cdot (inner)'$

$$h'(x) = f'(g(x)) \cdot g'(x)$$

g(x) doesn't change here

ex: $h(x) = \sqrt{x^2 + 1}$

inner	outer
$g = x^2 + 1$	$f = \sqrt{x}$
$g' = 2x$	$f' = \frac{1}{2\sqrt{x}}$

$$f'(g) = \frac{1}{2\sqrt{x^2 + 1}}$$

$$\therefore h'(x) = f'(g) \cdot g'$$

$$= \frac{1}{2\sqrt{x^2 + 1}} \cdot 2x$$

combine into single fraction

$$= \frac{x}{\sqrt{x^2 + 1}}$$

Chain Rule with *u*-Substitution

ex. differentiate $h(x) = (x^3 - 1)^{100}$

define inner function as u: $u = x^3 - 1$

$u' = 3x^2$

then $h = u^{100}$

and $h' = \left(u^{100}\right)' \cdot u'$

$= 100u^{99} \cdot u'$

convert back to x $= 100(x^3 - 1)^{99} \cdot 3x^2$

combine factors $= \boxed{300x^2(x^3 - 1)^{99}}$

re-visit: $h(x) = \sqrt{x^2 + 1}$

$u = x^2 + 1$

$u' = 2x$

$h' = \left(\sqrt{u}\right)' u'$

$= \dfrac{1}{2\sqrt{u}} u'$

$= \dfrac{1}{2\sqrt{x^2 + 1}} \cdot 2x = \boxed{\dfrac{x}{\sqrt{x^2 + 1}}}$

Chain Rule

$(outer)' \cdot (inner)'$

$h'(x) = f'(g(x)) \cdot g'(x)$

Chain Rule - examples

ex. find $\frac{dy}{dx}$ of $y = e^{2x}$

$$= e^u \quad \text{recall } \left(e^u\right)' = e^u$$

$$u = 2x$$

$$\frac{du}{dx} = 2$$

$$\frac{dy}{dx} = e^u \cdot \frac{du}{dx}$$

$$= \boxed{2e^{2x}}$$

Chain Rule

$(outer)' \cdot (inner)'$

$h'(x) = f'(g(x)) \cdot g'(x)$

Product Rule

$y' = f'g + fg'$

ex. find y' of $y = (1-x)^2 = u^2$

Chain Rule:

$u = 1 - x$

$u' = -1$

$$y' = 2u \cdot u'$$

$$= 2(1-x)\cdot(-1)$$

$$= \boxed{-2\,(1-x)}$$

Product Rule:

$f = (1-x) \quad g = (1-x)$

$f' = -1 \quad g' = -1$

$$y = \underset{f}{(1-x)}\underset{g}{(1-x)}$$

$$y' = -1(1-x) + (1-x)(-1)$$ combine like terms

$$\boxed{= -2(1-x)}$$

Practice

Do: find y' of $y = e^{-x}$

$= -e^{-x}$

Do: find y' of $y = e^{kx}$ where k is a constant

$= ke^{kx}$

Chain Rule

$(outer)' \cdot (inner)'$

$h'(x) = f'(g(x)) \cdot g'(x)$

More Chain Rule Examples

ex. find $f'(x)$ when $f(x) = \dfrac{1}{\sqrt[3]{x^2+x+1}}$ $= (x^2+x+1)^{-1/3}$

$$u = x^2 + x + 1$$

$$u' = 2x + 1$$

$$f'(x) = \left(u^{-1/3}\right)' \cdot u'$$

$$= -\frac{1}{3}u^{-4/3} \cdot u'$$

$$= -\frac{1}{3}(x^2+x+1)^{-4/3}(2x+1)$$

write as single fraction with a radical

$$= -\frac{2x+1}{3(x^2+x+1)^{+4/3}}$$

$$= -\frac{2x+1}{3\sqrt[3]{\left(x^2+x+1\right)^4}}$$

Chain Rule

$(outer)' \cdot (inner)'$

$$h'(x) = f'(g(x)) \cdot g'(x)$$

Chain Rule with Product Rule

product rule

ex. find y' of $y = (2x+1)^5 x^4$

$f \quad g$

chain rule

$f = \boxed{(2x+1)^5}$

$= u^5$

$u = 2x+1$

$u' = 2$

$g = \boxed{x^4}$

$g' = \boxed{4x^3}$

$f' = 5u^4 \cdot u'$

$= 5(2x+1)^4 \cdot 2$

$= \boxed{10(2x+1)^4}$

$f' \quad g \quad + \quad f \quad g'$

$y' = 10(2x+1)^4 \cdot x^4 + (2x+1)^5 \cdot 4x^3$

$= \boxed{10x^4(2x+1)^4 + 4x^3(2x+1)^5}$

Chain Rule with Quotient Rule

ex: find derivative of $g(t)=\left(\frac{t-2}{2t+1}\right)^9$

chain rule

$u=\frac{t-2}{2t+1}$ $\begin{matrix} f \\ g \end{matrix}$

quotient rule

$f=t-2$ $g=2t+1$

$f'=1$ $g'=2$

$g(t)=u^9$

$\therefore g'(t)=\left(u^9\right)' \cdot u'$

$=9u^8 \cdot u'$

$=9\left(\frac{t-2}{2t+1}\right)^8 \cdot \frac{5}{(2t+1)^2}$

$\left(\frac{a}{b}\right)^x=\frac{a^x}{b^x}$

$=9\,\frac{(t-2)^8}{(2t+1)^8} \cdot \frac{5}{(2t+1)^2}$

like bases

$=\boxed{\frac{45(t-2)^8}{(2t+1)^{10}}}$

$\therefore u'=\frac{(1)(2t+1)-2(t-2)}{(2t+1)^2}$

$=\frac{2t+1-2t+4}{(2t+1)^2}$

$u'=\frac{5}{(2t+1)^2}$